Appendix B

Effective theories of quantum Hall states and edge state transport

B.1 Chern-Simons gauge field

B.1.1 Chern-Simons action

First let us quickly look at what Chern-Simons action is and what it does. A gauge invariant term in $(2+1)$D is Maxwell term containing $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$.

$$S_{MX} = \int d^3x f_{\mu\nu} f_{\mu\nu} \tag{B.1}$$

Another example is so-called Chern-Simons term ($\epsilon_{\mu\nu\lambda}$ is the totally anti-symmetric tensor.)

$$S_{CS} = \int d^3x \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \tag{B.2}$$

The Hamiltonian of the Chern-Simons field in the gauge of $a_0 = 0$ vanishes

$$H_{CS} = \int d^3x a_0 \epsilon^{ij} \partial_i a_j = 0 \tag{B.3}$$

Therefore, the spectrum of the Chern-Simons gauge field is identically zero and costs no energy to populate them. The equation of motion of the Chern-Simons field is

$$\frac{\delta S_{CS}}{\delta a_\mu} = f_{\nu\lambda} = 0 \tag{B.4}$$

Namely, the field strength of the Chern-Simons field is zero. The solution is

$$a_\mu = \partial_\mu \chi \tag{B.5}$$

This is merely a gauge transformation. Hence, the Chern-Simons gauge field is not dynamical by itself. Let us couple the gauge field to a matter field $j_\mu = (\rho, j_i)$ and $\partial_\mu j_\mu = 0$.

$$\mathscr{L} = -\frac{\kappa}{4\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + a_\mu j_\mu \tag{B.6}$$

, where $\kappa > 0$ is an integer. The equation of motion of a_μ is

$$\frac{\partial \mathscr{L}}{\partial a_\mu} - \partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu a_\nu)} = j_\mu - \frac{\kappa}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda = 0 \tag{B.7}$$

, defining $f_{ij} = b$ for $\mu = 0$ and $f_{0i} = e_i$ for $\mu = 1, 2$,

$$b = \frac{2\pi}{\kappa}\rho \tag{B.8}$$

$$\epsilon_{ij}e_j = \frac{2\pi}{\kappa}j_i \tag{B.9}$$

Apparently the dynamics of the Chern-Simons field is now induced upon the coupling to the matter field. Regarding that a_0 does not have a time derivative term, one can view $\mu = 0$ component as a constraint between the matter field and the Chern-Simons gauge field: flux attachment $\frac{2\pi}{\kappa}\rho = b$ attaching $\frac{2\pi}{\kappa}$ gauge flux of b to the particle. This is the heart of the Chern-Simons action in the context of quantum Hall effect. As will be seen, dressing the matter with Chern-Simons gauge field can induce fractional statistics on the particles of the matter.

B.1.2 Flux attachment transformation

In the following, the key point will be so-called Chern-Simons gauge field coupled to matter field. Hence let us illuminate the operation of the flux attachment. Suppose there is a free (2+1)D fermionic field under magnetic field $\vec{A}$

$$\mathscr{L} = \psi^*(i\partial_0 - qA_0)\psi - \frac{1}{2m}\psi^*(i\partial_i - qA_i)^2\psi \tag{B.10}$$

, where $A_0 = 0$ and ψ satisfies the anti-commutation relation $\left[\psi(\vec{x}), \psi^\dagger(\vec{x'})\right]_+ = \delta(\vec{x} - \vec{x}')$. If we can map the problem of particles under magnetic field to some different particles under different magnetic field, we may able to solve the problem or obtain a better picture of the system. Hence we wish to attach $p\Phi_0$ of fictitious magnetic fluxes to a particle at position $\vec{x}_0$, where $\Phi_0 = \frac{h}{e} = 2\pi$ in the unit of $\hbar = e = 1$. Then the vector potential $\vec{a}$ that generates such a flux is as follows.

$$p\Phi_0 \delta(\vec{x} - \vec{x}_0) = \nabla \times \vec{a}(\vec{x}) \tag{B.11}$$

$$\vec{a}(\vec{x}) = \frac{p\Phi_0}{2\pi} \nabla\theta(\vec{x} - \vec{x}_0) = \frac{p\Phi_0}{2\pi} \frac{\vec{\hat{z}} \times (\vec{x} - \vec{x}_0)}{|\vec{x} - \vec{x}_0|^2} \tag{B.12}$$

,where $\theta(\vec{x} - \vec{x}_0)$ is the angle of $\vec{x}_0$ looking from $\vec{x}$ in counter-clockwise. Extending this to many particle system, the particle density $\rho = \psi^\dagger\psi$ is $\rho(\vec{x}) = \sum_i \delta(\vec{x} - \vec{x}_i)$ and the vector potential is

$$\vec{a}(\vec{x}) = \sum_i \frac{p\Phi_0}{2\pi} \nabla\theta(\vec{x} - \vec{x}_0) = \frac{p\Phi_0}{2\pi} \int dx'^3 \rho(\vec{x}') \nabla\theta(\vec{x} - \vec{x}') \tag{B.13}$$

Now the constraint reads

$$p\Phi_0 \rho(\vec{x}) = \epsilon_{ij} \partial_i a_j(\vec{x}) \qquad (i, j = 1, 2) \tag{B.14}$$

, where $\epsilon_{ij} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. The fluxes are locked to the particles at all time. One can rephrase it as a constraint between the fictitious magnetic flux and particles. Expressing a constraint $p\Phi_0\rho = \epsilon_{ij}\partial_i a_j$ is best done in the Lagrangian formalism. Employing a flux-attachment constraint term L_{fa} with a Lagrange multiplier $a_0 = 0$ as

$$\mathscr{L}_{fa} = a_0 \left(\frac{1}{p\Phi_0} \epsilon_{ij} \partial_i a_j - \rho \right) \tag{B.15}$$

merely adding L_{fa} to the original Lagrangian does the flux attachment.

$$\mathscr{L} = \psi^*(i\partial_0 - qA_0)\psi - \frac{1}{2m}\psi^*(i\partial_i - qA_i)^2\psi + a_0 \left(\frac{1}{p\Phi_0} \epsilon_{ij} \partial_i a_j - \rho \right) \tag{B.16}$$

Now we perform a gauge transformation $\psi = e^{iJ(\vec{x})}\phi$ such that

$$J(\vec{x}) = -p \int dx'^3 \rho(\vec{x}') \theta(\vec{x} - \vec{x}') \quad , \quad a_i(\vec{x}) = \frac{\Phi_0}{2\pi} \partial_i J(\vec{x}) \tag{B.17}$$

Substituting $\psi = e^{iJ(\vec{x})}\phi$ and the Lagrangian is now

$$\mathscr{L} = \phi^*\big(i\partial_0 - (qA_0 + a_0)\big)\phi - \frac{1}{2m}\phi^*\big(i\partial_i - (qA_i + a_i)\big)^2\phi + \frac{1}{p\Phi_0}a_0\epsilon_{ij}\partial_i a_j \tag{B.18}$$

Note that $A_0 = a_0 = 0$. Important to notice that the commutation relation of ϕ is

$$\big[\phi(\vec{x}), \phi^\dagger(\vec{x'})\big]_+ = \delta(\vec{x} - \vec{x}') \quad , \quad (p : even) \tag{B.19}$$

$$\big[\phi(\vec{x}), \phi^\dagger(\vec{x'})\big]_- = \delta(\vec{x} - \vec{x}') \quad , \quad (p : odd) \tag{B.20}$$

Therefore, attaching the fictitious magnetic flux to the particle changes the statistics of the particles. If the constraint does not exist, nothing forces the fluxes to follow the motion of the particles. Hence one can not define the angles between the particles and the gauge transformation by $\psi = e^{iJ(\vec{x})}\phi$ becomes meaningless. For odd p, ϕ becomes bosonic and ϕ stays fermionic for even p. If the strength of the external magnetic field is $-p$ times the electron density, the fictitious magnetic field will cancel the external magnetic field on the average and ϕ becomes bosons at zero effective magnetic field. The ground state of such system will be bose-condensed state (superconducting state). Note that the argument so far did not mention anything about Coulomb interaction among particles which is supposed to be the crucial ingredient for the FQHE. Now we should remember that the angle between particles cannot be defined if they stand on the same location. Hence, the validity of the flux attachment actually relies on the strong Coulomb interaction among the particles so that they would not come to the same position. Though we will not include the interaction among particles in the following argument, one should recognize their role. After all, what we derive here is the long-distance/time effective theory of fractional quantum Hall fluids. If one wishes to know various (fast) dynamics and electromagnetic response of them, the details of interaction should be considered.

In the general gauge ($\Phi_0 = 2\pi$)

$$S_{CS} = \frac{1}{4\pi}\frac{1}{p}\int d^3x\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \tag{B.21}$$

that transforms as $a_\mu \to a_\mu + \partial_\mu\chi$. As will be seen, there are two kinds of Chern-Simons field in the effective theory of quantum Hall states. One is this Chern-Simons field for the flux attachment transformation, which is crucial for a rigorous derivation of the effective theory but does not have to show up in a heuristic derivation (see later). The other, which will appear repeatedly in the following, appears upon a so-called duality transformation between particles and vortices. The reader should notice the distinction between their origins since they seem superficially indifferent.

B.1.3 Duality transformation

Here we introduce so-called duality transformation in the context of quantum Hall effect, which exchange particles and vortices (quasiparticles). In the dual picture, the vortices become the fundamental particles and the particles become the vortices. The physical origin of the transformation traces back to the fact that winding a vortex (quasiparticle) around a particle and vice verse but in the other way around do the same job in two dimension. Namely one can describe the same system focusing on the particles as well as the vortices. In general gauge, the flux-attached Lagrangian is

$$\mathscr{L} \;=\; \phi^*\big(i\partial_0 - (qA_0 + a_0)\big)\phi - \frac{1}{2m}\phi^*\big(i\partial_i - (qA_i + a_i)\big)^2\phi + \frac{1}{4\pi}\frac{1}{p}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$$

In the condensed phase of ϕ by choosing odd p properly, $\phi = \sqrt{\rho}e^{e\theta}$ and the low-energy action is as follows. The first term describes the phase stiffness.

$$\mathscr{L} \;=\; \frac{1}{2}\rho\big(\partial_\mu\theta - (qA_\mu + a_\mu)\big)^2 + \frac{1}{4\pi}\frac{1}{p}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \tag{B.23}$$

The gauge invariant ϕ particle current is $j_\mu = \frac{\delta S}{\delta\partial_\mu\theta}$.

$$j_\mu \;=\; \rho\big(\partial_\mu\theta - (qA_\mu + a_\mu)\big) \tag{B.24}$$

j_μ satisfies the continuity equation $\partial_\mu j_\mu = 0$ or equivalently $div j_\mu = 0$, which can be solved simply, reminding $div \cdot rot = 0$,

$$j_\mu \;=\; \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda \tag{B.25}$$

This re-expression of j_μ is the heart of duality transformation. Since the excitation of the condensate is not only the phase modulation but also amplitude modulation that leads to topological excitaitons (vortices). The phase of the condensate winds by 2π around one vortex. The vortex (quasiparticle in QHE) j_μ^{qp} in the condensate can be described by

$$j_\mu^{qp} = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda\theta \tag{B.26}$$

One can see the equivalence to $\nabla \times a_\mu$ from the previous section. Substituting j_μ^{qp} and the dual j_μ back into the original j_μ,

$$\frac{1}{\rho}\partial_\nu f_{\mu\nu} = j_\mu^{qp} - \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\big(qA_\lambda + a_\lambda\big) \tag{B.27}$$

, where $f_{\mu\nu} = \partial_\mu\alpha_\nu - \partial_\nu\alpha_\mu$. This equation of motion for α_μ is derived from dual Lagrangian

$$\mathscr{L}_{dual} = \frac{1}{2\rho}f_{\mu\nu}f_{\mu\nu} + \alpha_\mu j_\mu^{qp} + \frac{q}{2\pi}\alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda + \frac{1}{2\pi}\alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda + \frac{1}{4\pi}\frac{1}{p}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda \tag{B.28}$$

From this Lagrangian, we want to eliminate a_μ so that we can look only at the dynamics of j_μ^{qp} and α_μ coupled to electromagnetic field. Using $(\epsilon\partial)_{\mu\lambda} \equiv \epsilon^{\mu\nu\lambda}\partial_\nu$, let us make a shift $a_\mu = \tilde{a}_\mu + p\alpha_\mu$ and drop (integrate out) $\tilde{a}_\mu$.

$$\mathscr{L}_{dual} = \frac{1}{2\rho}f_{\mu\nu}f_{\mu\nu} + \alpha_\mu j_\mu^{qp} + \frac{q}{2\pi}\alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda - \frac{p}{4\pi}\alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda \tag{B.29}$$

If we only look at the long wavelength and low energy part of the dynamics, we can neglect the Maxwell term because it has two derivatives while the Chern-Simons term has only one derivative.

$$\mathscr{L}_{dual} = -\frac{p}{4\pi}\alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda + \alpha_\mu j_\mu^{qp} + \frac{q}{2\pi}\alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda \tag{B.30}$$

We have obtained the dual Lagrangian where the quasiparticles (vortices) act as the fundamental particles interacting with the gauge field that is the dual representation of the original particles. One can view the vortices feels the Magnus force from the condensate of the particles.

B.1.4 Fractional statistics

Suppose $A_\mu = 0$ and let us derive the effective action for the matter field. Shift α_μ such that $\alpha_\mu = \tilde{\alpha}_\mu + \frac{2\pi}{p}(\epsilon\partial)_{\mu\lambda}j_\lambda^{qp}$.

$$S = \int d^3x\Big[-\frac{p}{4\pi}\tilde{\alpha}_\mu(\epsilon\partial)_{\mu\lambda}\tilde{\alpha}_\lambda + \frac{\pi}{p}j_\mu^{qp}(\epsilon\partial)^{-1}_{\mu\lambda}j_\lambda^{qp}\Big] \tag{B.31}$$

Here we dropped the suffix "dual". Now integrating out $\tilde{\alpha}_\mu$ is easy and we are left with only the dressed quasiparticles.

$$S = \frac{\pi}{p}\int d^3x j_\mu^{qp}(\epsilon\partial)^{-1}_{\mu\lambda} j_\lambda^{qp} \tag{B.32}$$

We would like to consider the phase acquired upon encircling particle 1 around particle 2. The two particle system is $j_\mu^{qp} = j_{\mu,1}^{qp} + j_{\mu,2}^{qp}$, where $j_{0,n}^{qp} = l_n\delta(\vec{x}-\vec{x}_n(t))$ and $j_{i,n}^{qp} = l_n\dot{x}_{i,n}\delta(x-\vec{x}_n(t))$. Let us introduce l_1 and l_2 as the number of the agglomeration of the paritcles at a location. Fixing the particle 2 $\vec{x}_2(t) = 0$, $j_{0,2}^{qp} = l_2\delta(x)$ and $j_{i,2}^{qp} = 0$. Define $(\epsilon\partial)^{-1}_{\mu\lambda} j_{\lambda,2}^{qp} \equiv f_\mu$,

$$\epsilon_{0\nu\lambda}\partial_\nu f_\lambda = \delta(x) \quad , \quad \vec{f}(\vec{x}) = \frac{1}{2\pi}\frac{\vec{z}\times\vec{x}}{|\vec{x}|^2} \tag{B.33}$$

f_μ is the vector potential of a delta-like flux of the gauge field ($f_0 = 0$). We can calculate the Berry phase. The trajectory of the particle 1 is $\vec{x}_1(t) = r(cos\theta, sin\theta)$ and $\dot{\vec{x}}_1(t) = \dot{\theta}r(-sin\theta, cos\theta)$ for $0 \leq \theta \leq 2\pi$, which is equivalent to exchanging the particles twice and hence aquire twice the exchange statistical phase $2\theta_{CS} = \frac{\pi l_1 l_2}{p}\int d^3x j_{\mu,1}^{qp} f_\mu = \frac{l_1 l_2}{p}\int dt\dot{\theta}$.

$$\theta_{CS} = \frac{\pi l_1 l_2}{p} \tag{B.34}$$

Therefore, the quasiparticles dressed by α_μ act as carrying $\frac{1}{p}$ of ϕ. For example, for $l_1 = l_2 = 1$ and $p = 3$ (filling factor $\nu = 1/3$), $\theta_{CS} = \frac{\pi}{3}$. A bunch of three particles, $l_1 = l_2 = 3$, leads to $\theta_{CS} = 3\pi \to \pi$, recovering the fermionic statistics.

B.1.5 Gauge transformation and edge action

Assuming $j_\mu = 0$, let us performe a gauge transformation $\alpha_\mu \to \alpha_\mu + \partial_\mu\chi$,

$$\delta S_{CS} = -\frac{p}{4\pi}\int d^3x\epsilon_{\mu\nu\lambda}\partial_\mu\chi\partial_\nu\alpha_\lambda = -\frac{p}{4\pi}\int d^3x\partial_\mu\big(\chi\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda\big) \tag{B.35}$$

The total derivative can be transformed to the boundary integral, which vanishes in systems without boundaries. The gauge invariance therein is preserved. However, for systems with boundary, the boundary term does not vanish and S_{CS} breaks the gauge invariance.

Note that if we substitute the solution of the Euler-Lagrange equation, the classical trajectory of the field, $\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda = 0$ into the boundary term yields $\delta S_{CS} = 0$. Therefore, the Chern-Simons action is gauge invariant in the classical theory. However, it breaks down in the quantum theory and demands a physical degree of freedom to preserve the gauge invariance. We have two ways to remedy this problem; (1) make the boundary physical by restricting the gauge or (2) add an action cancelling the boundary term. Here, we take (1) and set the boundary along x at $y = 0$ (unbounded in t and x)

$$\delta S_{CS} = -\frac{p}{4\pi}\int dxdt\chi\epsilon_{y\nu\lambda}\partial_\nu\alpha_\lambda \tag{B.36}$$

Here we again have two ways to maintain the gauge symmetry $\delta S_{CS} = 0$; (1) $\chi|_{y=0} = 0$ or (2) $\epsilon_{y\nu\lambda}\partial_\nu\alpha_\lambda = 0$. Here we take (1). The condition $\chi|_{y=0} = 0$ means that once a gauge is chosen, one cannot change the gauge freely any more. What one can still do is to take a different gauge function $\tilde{\chi} = \chi + c$, where $c|_{y=0} = 0$. Even for a gauge potential of the form $\alpha_\mu = \partial_\mu\phi$, the condition $\chi|_{y=0} = 0$ forbids to gauge it away, which makes ϕ dynamical. In other words, different gauges for α_μ correspond to different physical realizations of the boundary and one cannot connect them by gauge transformations. Here, the different physical realizations mean different edge velocities v, which is determined by the steepness of the confinement pontential setting the boundary of the system. For now, let us here take $\alpha_0 = 0$ as the gauge condition, which leads to the constraint $\epsilon_{ij}\partial_i\alpha_j = 0$ for $i, j = 1, 2$. Its solution is $\alpha_i = \partial_i\phi'$ and substituting them into the action leads to

$$S_{CS} = -\frac{p}{4\pi}\int d^3x\partial_y(\partial_x\phi'\partial_t\phi') = -\frac{p}{4\pi}\int dtdx\partial_t\phi'\partial_x\phi'|_{y=0} \tag{B.37}$$

Therefore, the edge action is, introducing $\phi'(x, y, t)|_{y=0} = \phi(x, t)$,

$$S_{edge} = -\frac{p}{4\pi}\int dtdx\partial_t\phi\partial_x\phi \tag{B.38}$$

, whose Hamiltonian is indentically zero.

$$H_{edge} = 0 \tag{B.39}$$

Something is apparently wrong here. Now let us change the coordinate of the system such that $\tilde{t} = t$, $\tilde{x} = x - vt$ and $\tilde{y} = y$. Suppose $\phi(x, y, t)$ transforms to $\tilde{\phi}(\tilde{x}, \tilde{y}, \tilde{t})$

by the coordinate change. $\frac{\partial\tilde{\phi}(\tilde{x},\tilde{t})}{\partial x} = \frac{\partial\tilde{x}}{\partial x}\frac{\partial\tilde{\phi}(\tilde{x},\tilde{t})}{\partial\tilde{x}} + \frac{\partial\tilde{t}}{\partial x}\frac{\partial\tilde{\phi}(\tilde{x},\tilde{t})}{\partial\tilde{t}} = \frac{\partial\tilde{\phi}(\tilde{x},\tilde{t})}{\partial\tilde{x}}$. Similarly, $\frac{\partial\tilde{\phi}(\tilde{x},\tilde{t})}{\partial t} = \frac{\partial\tilde{x}}{\partial t}\frac{\partial\tilde{\phi}(\tilde{x},\tilde{t})}{\partial\tilde{x}} + \frac{\partial\tilde{t}}{\partial t}\frac{\partial\tilde{\phi}(\tilde{x},\tilde{t})}{\partial\tilde{t}} = -v\frac{\partial\tilde{\phi}(\tilde{x},\tilde{t})}{\partial\tilde{x}} + \frac{\partial\tilde{\phi}(\tilde{x},\tilde{t})}{\partial\tilde{t}}$, which reads $\partial_{\tilde{t}} = \partial_t + v\partial_x$, $\partial_{\tilde{x}} = \partial_x$ and $\partial_{\tilde{y}} = \partial_y$. Therefore, $\alpha_{\tilde{t}} = \alpha_t + v\alpha_x$, $\alpha_{\tilde{x}} = \alpha_x$ and $\alpha_{\tilde{y}} = \alpha_y$. Taking the new constraint $\alpha_{\tilde{t}} = 0$ leads to $S_{edge} = \frac{p}{4\pi}\int d\tilde{t}d\tilde{x}\partial_{\tilde{t}}\tilde{\phi}\partial_x\tilde{\phi}$.

$$S_{edge} = -\frac{p}{4\pi}\int dtdx(\partial_t + v\partial_x)\phi\partial_x\phi \tag{B.40}$$

, whose Hamiltonian is

$$H_{edge} = \frac{pv}{4\pi}\int dtdx\big(\partial_x\phi\big)^2 \tag{B.41}$$

For the stability, $p > 0$ and $v > 0$. The Chern-Simons theory is known to be a metric-free topological field theory, hence the bulk theory is independent of the choice of the metric. But the action of the edge depends on the specific metric. It is evident that the change of coordinate cannot be done by the gauge transformations.

B.2 Bulk effective theory of quantum Hall liquid

B.2.1 Heuristic derivation of hierarchical bulk action

Here, we discuss the heuristic argument that leads to the effective action of the bulk S by Wen and Zee based on the hierarchical scheme. First, let us consider principal Laughlin states $\nu = 1/m$, where m is an odd number. There are several conditions that the effective action should satisfy. Among them, showing here the current conservation and the experimentally observed Hall conductance,

$$(1) \;:\; \partial_\mu j_\mu = 0 \tag{B.42}$$

$$(2) \;:\; qj_\mu = \nu\frac{q^2}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda \tag{B.43}$$

, where $q = -e$ and $B < 0$, hence the Hall conductance is $-\nu\frac{q^2}{2\pi}$ and the electric field is $-\epsilon_{0\nu\lambda}\partial_\nu A_\lambda$. The condition (1) can be solved as follows.

$$j_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda \tag{B.44}$$

The electric current is derived by $\frac{\delta S}{\delta A_\mu} = qj_\mu = \frac{q}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda$. The condition (2) can be satisfied if one takes $\mathscr{L} = -\frac{1}{2\nu}\alpha_\mu j_\mu + qj_\mu A_\mu$. Computing the Euler-Lagrange equation for α_μ recovers the condition (2).

$$\mathscr{L} \;=\; -\frac{m}{4\pi}\alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda + \frac{q}{2\pi}A_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda \tag{B.45}$$

The quasiparticle excitation can be supplemented with a source term $l\alpha_\mu j^{qp}_\mu$, where l is the ”charge” representing the coupling strength between j^{qp}_μ and the quasiparticles, acting as bosons, only see the Chern-Simons gauge field as the ”magnetic” field. Since they are not directly coupled to the EM field, they carry no electric charge. If one integrate out α_μ, the dressed quasiparticles are directly coupled to the EM field and acquires the electric charge and the correct fractional statistics. When the density of quasiparticles become $j_0 = 2j^{qp}_0$ (2 flux quantum of α_μ per quasiparticle), the quasiparticles can bose-condense. The dual representation of the quasiparticle is $j^{qp}_\mu = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\tilde{\alpha}_\lambda$, hence the constraint reads $\frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda = \frac{2}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\tilde{\alpha}_\lambda$ with $\mu = 0$. Thus, the action of the second hierarchy is

$$\mathscr{L} \;=\; \frac{1}{2\pi}\alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\tilde{\alpha}_\lambda - \frac{2}{4\pi}\tilde{\alpha}_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\tilde{\alpha}_\lambda \tag{B.46}$$

, hence adding the second hierarchy to the first one, the full action is

$$\mathscr{L} \;=\; -\frac{1}{4\pi}K_{IJ}\alpha^I_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\alpha^J_\lambda + \frac{q}{2\pi}t_I A_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\alpha^I_\lambda \tag{B.47}$$

, where $q = -e$, $(\alpha^1, \alpha^2) = (\alpha, \tilde{\alpha})$, $(t_1, t_2) = (1, 0)$ and $K = \left(\begin{smallmatrix} p_1 & -1 \\ -1 & p_2 \end{smallmatrix}\right)$ with $p_1 = m$ and $p_2 = 2$. Here, the Einstein's summation convention is implied. For $m = 3$, the action describes $\nu = 2/5$, namely, $K_{2/5} = \left(\begin{smallmatrix} 3 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. The third hierarchy describes $\nu = 3/7$ state, where $K_{3/7} = \left(\begin{smallmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{smallmatrix}\right)$.

Let us derive the filling factor $\nu = \frac{qB}{2\pi j^1_0}$ from the action in a incremental manner. The Euler-Lagrange equation of the time component α^1_0 and α^2_0

$$\frac{qB}{2\pi} \;=\; p_1 j^1_0 - j^2_0 \quad , \quad j^1_0 \;=\; p_2 j^2_0 \quad \Rightarrow \quad \nu = \frac{1}{p_1 - \frac{1}{p_2}} \tag{B.48}$$

$$\frac{qB}{2\pi} \;=\; p_1 j^1_0 - j^2_0 \quad , \quad j^1_0 \;=\; p_2 j^2_0 - j^3_0 \quad , \quad j^2_0 \;=\; p_3 j^3_0 \quad \Rightarrow \quad \nu = \frac{1}{p_1 - \frac{1}{p_2 - \frac{1}{p_3}}} \tag{B.49}$$

One can show that the filling factor can be derive as follows.

$$\mathbf{t}^\top K^{-1}\mathbf{t} \;=\; K^{-1}_{1,1} \;=\; \frac{1}{p_1 - \frac{1}{p_2 - \cdots}} \tag{B.50}$$

So far, the hierarchical states appeared above are only the particle-like states. Let us now consider a hole-like state $\nu = 2/3$. Having the first hierarchy $\nu = 1$, the inter-hierarchy coupling is $j_0^1 = -2j_0^2$, which comes from an action

$$\mathscr{L} \;=\; \frac{1}{2\pi}\alpha_\mu \epsilon_{\mu\nu\lambda}\partial_\nu \tilde{\alpha}_\lambda - \frac{-2}{4\pi}\tilde{\alpha}_\mu \epsilon_{\mu\nu\lambda}\partial_\nu \tilde{\alpha}_\lambda \tag{B.51}$$

Therefore, the K matrix of $\nu = 2/3$ stateis $K_{2/3} = \begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix}$. Just to make sure,

$$\frac{qB}{2\pi} \;=\; j_0^1 - j_0^2 \quad , \quad j_0^1 \;=\; -2j_0^2 \quad \Rightarrow \quad \nu = \frac{1}{1 - \frac{1}{-2}} \;=\; \frac{2}{3} \tag{B.52}$$

Likewise, the K matrix of $\nu = 3/5$ state is $K_{3/5} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & -2 & -1 \\ 0 & -1 & -2 \end{pmatrix}$, which indeed yields the correct filling factor.

B.2.2 Rigorous derivation of hierarchical bulk action

Let us recall the dual Lagrangian after integrating out the gauge field $a_\mu \equiv a_\mu^1$ attaching p gauge flux to the matter and assign proper suffices for convenience.

$$\mathscr{L} \;=\; \alpha_\mu^1 j_\mu^{qp1} + \frac{q}{2\pi}\alpha_\mu^1 \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda - \frac{p_1}{4\pi}\alpha_\mu^1 \epsilon_{\mu\nu\lambda}\partial_\nu \alpha_\lambda^1 \tag{B.53}$$

Just for clarification, we recover the j^{qp1} field explicitly.

$$\alpha_\mu^1 j_\mu^{qp1} \;\Leftarrow\; \phi_{qp1}^*\big(i\partial_0 - \alpha_0\big)\phi_{qp1} - \frac{1}{2m_1}\phi_{qp1}^*\big(i\partial_i - \alpha_i\big)^2\phi_{qp1} \tag{B.54}$$

If j^{qp1} is macroscopically populated that they can condense, we can do the same trick to "cancel" the gauge flux of α_μ^1 by attaching p_2 gauge flux of them via a_μ^2. One can see the correspondences $A_\mu \Rightarrow \alpha_\mu^1$ and $a_\mu^1 \Rightarrow a_\mu^2$ from the previous argument.

$$\mathscr{L} \;=\; \phi_{qp1}^*\big(i\partial_0 - (\alpha_0 + a_0^2)\big)\phi_{qp1} + \frac{q}{2m_1}\phi_{qp1}^*\big(i\partial_i - (\alpha_i + a_i^2)\big)^2\phi_{qp1} \tag{B.55}$$

$$+\frac{1}{4\pi p_2}a_\mu^2 \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda^2 - \frac{1}{2\pi}\alpha_\mu^1 \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda - \frac{p_1}{4\pi}\alpha_\mu^1 \epsilon_{\mu\nu\lambda}\partial_\nu \alpha_\lambda^1 \tag{B.56}$$

In the codensed phase of ϕ_{qp1}, $\phi_{qp1} = \sqrt{\rho_1}e^{i\theta_1}$,

$$\mathscr{L} = \frac{1}{2}\rho_1\big(\partial_\mu\theta_1 - (\alpha^1_\mu + a^2_\mu)\big)^2 + \frac{1}{4\pi p_2}a^2_\mu\epsilon_{\mu\nu\lambda}\partial_\nu a^2_\lambda + \frac{q}{2\pi}\alpha^1_\mu\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda - \frac{p_1}{4\pi}\alpha^1_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\alpha^1_\lambda \tag{B.57}$$

Now the dual representation of ϕ_{qp1} is

$$\rho_1\big(\partial_\mu\theta_1 - (\alpha^1_\mu + a^2_\mu)\big) = j^{qp1} = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\alpha^2_\lambda \tag{B.58}$$

In the same manner as before, integrating out a^2_μ from the action and neglecting the Maxwell term of α^2_λ leads to

$$\mathscr{L} = -\frac{p_1}{4\pi}\alpha^1_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\alpha^1_\lambda + \frac{1}{2\pi}\alpha^1_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\alpha^2_\lambda - \frac{p_2}{4\pi}\alpha^2_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\alpha^2_\lambda + \frac{q}{2\pi}\alpha^1_\mu\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$$

This can be encapslated into the aforementioned K-matrix of the second hierarchy $K = \left(\begin{smallmatrix} p_1 & -1 \\ -1 & p_2 \end{smallmatrix}\right)$ and $t = (1, 0)$.

B.2.3 Jain's construction

The Jain scheme is to view the FQHEs as an extension of IQHEs, where composite fermion Landau levels take place of the electron Landau levels by regarding an electron as a composite of a fermion with p flux quantum (p: even) in the opposite direction to the external magnetic field. Since p is an even number, it does not affect the statistics of the fermion upon an exchange with another. Let us first formulate the effective action for IQHE $\nu = N$. Each filled Landau level is the condensate of the composite boson with $m = 1$ viewed in the hierarchical scheme. Then, we can extend the previous scheme to incorporate the IQHEs, which is simply $K = 1_{N\times N}$ ($N \times N$ unit matrix) and $t = (1, 1, \cdots, 1)_N$. This is nothing but stacking N layers of $m = 1$ state in the hierarchical scheme.

$$\mathscr{L} = \psi^*(i\partial_0 - qA_0)\psi - \frac{1}{2m}\psi^*(i\partial_i - qA_i)^2\psi \tag{B.60}$$

Now, perform a transformation that regards ψ as a composite fermion $\psi_{CF} = e^{-iJ(x)}\psi$ attached with p (even number) gauge flux quantum of a_μ to partially cancel the external magnetic field.

$$\mathscr{L} = \psi^*_{CF}\big(i\partial_0 - (qA_0 + a_0)\big)\psi_{CF} - \frac{1}{2m}\psi^*_{CF}(i\partial_i - (qA_i + a_i))^2\psi_{CF} + \frac{1}{4\pi p}a_\mu\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda \tag{B.61}$$

Now suppose ψ_{CF} fill N composite fermion Landau levels and the gaps between the levels are big enough that each level conserve the number of fermions ψ_I.

$$\mathscr{L} = \sum_I \left[\psi_I^*\big(i\partial_0 - (qA_0 + a_0)\big)\psi_I - \frac{1}{2m}\psi_I^*(i\partial_i - (qA_i + a_i))^2\psi_I\right] + \frac{1}{4\pi p}a_\mu\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda \tag{B.62}$$

Now we do another transformation to view $\psi_I = e^{iJ_I(x)}\phi_I$ via a_μ^I.

$$\begin{aligned}\mathscr{L} = \sum_I \Big[&\phi_I^*\big(i\partial_0 - (qA_0 + a_0 + a_0^I)\big)\phi_I - \frac{1}{2m}\phi_I^*(i\partial_i - (qA_i + a_i + a_i^I)^2\phi_I \\ &+ \frac{1}{4\pi}a_\mu^I\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda^I\Big] + \frac{1}{4\pi p}a_\mu\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda\end{aligned} \tag{B.63}$$

In the condensed phase of ϕ_Is, the low-energy effective Lagrangian becomes

$$\mathscr{L} = \sum_I \left[\frac{\rho_I}{2}\big(\partial_\mu\theta - (qA_0 + a_0 + a_0^I)\big)^2 + \frac{1}{4\pi}a_\mu^I\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda^I\right] + \frac{1}{4\pi p}a_\mu\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda$$

Perform a duality transformation for each field

$$\rho_I\big(\partial_\mu\theta - (qA_0 + a_0 + a_0^I)\big) = j_{qpI} = \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda^I \tag{B.64}$$

The dual Lagrangian neglecting the Maxwell term,

$$\mathscr{L} = \frac{1}{2\pi}\left(\sum_I \alpha_\mu^I\right)\epsilon_{\mu\nu\lambda}\partial_\nu\big(a_\lambda + qA_\lambda\big) + \frac{1}{2\pi}\alpha_\mu^I\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda^I + \frac{1}{4\pi}a_\mu^I\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda^I + \frac{1}{4\pi p}a_\mu\epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda \tag{B.65}$$

Integrating out a_μ and a_I,

$$\mathscr{L} = \sum_{I=1}^N -\frac{1}{4\pi}\alpha_\mu^I\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda^I - \frac{p}{4\pi}\left(\sum_I \alpha_\mu^I\right)\epsilon_{\mu\nu\lambda}\partial_\nu\left(\sum_I \alpha_\mu^I\right) + \frac{q}{2\pi}A_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda^I$$

The K matrix is now

$$K = 1_{N\times N} + pC_{N\times N} \tag{B.67}$$

, where p is even, $C_{N\times N}$ is a matrix with 1 in all the components. The charge vector is $t = (1, 1, \cdots, 1)_N$. In the Jain's composite fermion picture, $\nu = 2/5$ can be regarded as the composite fermion filling factor 2 ($\nu_{CF} = 2$), where $N = 2$ and $p = 2$. Therefore, $K_{2/5} = \left(\begin{smallmatrix}3 & 2\\ 2 & 3\end{smallmatrix}\right)$. $\nu = 2/3$ is given $N = 2$ and $p = -2$. $K_{2/3} = \left(\begin{smallmatrix}1 & 2\\ 2 & 1\end{smallmatrix}\right)$ with $t = (1, 1)$ for both.

B.2.4 Filling factor, quasiparticle charge and statistics

Be it hierarchical scheme or Jain scheme, the effective Lagrangian of fractional quantum Hall effects with some populations of quasiparticles is compactly described as follows.

$$\mathscr{L} = -\frac{1}{4\pi}K_{IJ}\epsilon^{\mu\nu\lambda}\alpha^I_\mu\partial_\nu\alpha^J_\lambda + \frac{q}{2\pi}t_I\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu\alpha^I_\lambda + l_I\alpha^I_\mu j^{qpI}_\mu \tag{B.68}$$

we now wish to integrate out the Chern-Simons gauge fields in order to look only at long distance physics of the electromagnetic response and the quasiparticles of quantum Hall fluids. The Euler-Lagrange equation with respect to α^I_μ yields

$$\partial_\lambda\frac{\delta\mathscr{L}}{\delta(\partial_\lambda\alpha^I_\mu)} - \frac{\delta\mathscr{L}}{\delta\alpha^I_\mu} = \frac{1}{2\pi}K_{IJ}(\epsilon\partial)_{\mu\lambda}\alpha^J_\lambda - \frac{q}{2\pi}t_I(\epsilon\partial)_{\mu\lambda}A_\lambda - l_I j^{qpI}_\mu = 0 \tag{B.69}$$

$\mu = 0$ component reads

$$K_{IJ}j^J_0 = t_I\frac{qB}{2\pi} + l_I j^{qpI}_0 \tag{B.70}$$

Performing a shift for $\alpha^I_\mu = \hat{\alpha}^I_\mu + \tilde{\alpha}^I_\mu$, where $\alpha^I_\mu = 2\pi K^{-1}_{IJ}\big(\frac{q}{2\pi}t_J A_\mu + (\epsilon\partial)^{-1}_{\mu\lambda}l_J j^{qpJ}_\lambda\big)$ and substituting the shifted gauge fields into the original Lagrangian,

$$\mathscr{L} = -\frac{1}{4\pi}K_{IJ}\hat{\alpha}^I_\mu(\epsilon\partial)_{\mu\lambda}\hat{\alpha}^J_\lambda + \frac{1}{4\pi}K_{IJ}\tilde{\alpha}^I_\mu(\epsilon\partial)_{\mu\lambda}\tilde{\alpha}^J_\lambda \tag{B.71}$$

$\hat{\alpha}^I_\mu$ and $\tilde{\alpha}^I_\mu$ are decoupled, hence we can easily integrate out all the $\hat{\alpha}^I_\mu$s. Explicitly calculating the second term,

$$\begin{aligned}\mathscr{L}_{eff} &= \frac{q^2}{4\pi}t_I K^{-1}_{IJ}t_J A_\mu(\epsilon\partial)_{\mu\lambda}A_\lambda + qt_I K^{-1}_{IJ}l_J j^{qpJ}_\mu A_\mu + \pi l_I K^{-1}_{IJ}l_J j^{qpI}_\mu(\epsilon\partial)^{-1}_{\mu\lambda}j^{qpJ}_\lambda \\ &\equiv \nu_C\frac{q^2}{4\pi}A_\mu(\epsilon\partial)_{\mu\lambda}A_\lambda + q_J j^{qpJ}_\mu A_\mu + \theta_{IJ}j^{qpI}_\mu(\epsilon\partial)^{-1}_{\mu\lambda}j^{qpJ}_\lambda\end{aligned} \tag{B.72}$$

, where ν_C is filling factor ($\sigma_{xy} = \nu_C\frac{e^2}{2\pi\hbar}$), q_J is the charge of the quasiparticles specie J ($q = -e$) and θ_{IJ} is the statistical angle between quasiparticle species I and J. This describes the long-distance electromagnetic response of quantum Hall states.

$$\nu_C = \sum_{I,J}t_I K^{-1}_{IJ}t_J \tag{B.73}$$

$$q_J = q\sum_I t_I K^{-1}_{IJ}l_J \tag{B.74}$$

$$\theta_{IJ} = \pi l_I K^{-1}_{IJ}l_J \tag{B.75}$$

B.3 Edge state transport

B.3.1 Reduction from bulk to edge

Performing a gauge transformation for a system with boundary leads to a violation of gauge symmetry. Taking S and $\tilde{a}^I_\mu = a^I_\mu + \partial_\mu f^I$ (Note that a^I_μ here is different from a^I_μ appeared previously),

$$\delta S = \frac{1}{4\pi} K_{IJ} \int dx^3 \epsilon^{\mu\nu\lambda} \partial_\nu (f^I \partial_\mu a^I_\lambda) \tag{B.76}$$

The edge action reads

$$S_{edge} = \int dtdx \Big[-\frac{1}{4\pi} \big(K_{IJ} \partial_t \phi^I \partial_x \phi^J + V_{IJ} \partial_x \phi^I \partial_x \phi^J \big) + \frac{qt_I}{2\pi} A_\mu \epsilon_{\mu\nu} \partial_\nu \phi^I \Big] \tag{B.77}$$

The third term is $\int dxdt A_\mu J_\mu$ with $\mu = 0, 1$ and we drop it for simplicity.

$$S_{edge} = -\frac{1}{4\pi} \int dtdx K_{IJ} \partial_t \phi^I \partial_x \phi^J + V_{IJ} \partial_x \phi^I \partial_x \phi^J \tag{B.78}$$

Having the canonical momentum of ϕ^I, $\frac{\delta \mathscr{L}_{edge}}{\partial \partial_t \phi^I} = \partial_x \phi^I$, the Hamiltonian is

$$H_{edge} = \frac{1}{4\pi} \int dx V_{IJ} \partial_x \phi^I \partial_x \phi^J \tag{B.79}$$

Therefore, in order for the energy to be positive, V_{IJ} must be positive definite.

B.3.2 Quantized thermal conductance

Let us diagonalize K_{IJ} and V_{IJ} simultaneously in three steps. Both matrices are real and symmetric. Similarity transformation by Λ_1 with $\Lambda_1^\top \Lambda_1 = 1$, $\Lambda_1^\top K \Lambda_1 = \lambda_j \delta_{ij} \equiv K_D$ and then employing $\Lambda_2 = \delta_{ij}/\sqrt{|\lambda_j|}$ but with $\Lambda_2^\top \Lambda_2 = \delta_{ij}/|\lambda_j| = |K_D^{-1}|$, $\Lambda_2^\top \Lambda_1^\top K \Lambda_1 \Lambda_2 = sgn(\lambda_i)\delta_{ij} \equiv \eta_i \delta_{ij}$. Now, a diagonal matrix η with entries of ± 1 apparently commutes with $\Lambda_2^\top \Lambda_1^\top V \Lambda_1 \Lambda_2$, which is real and symmteric still. Hence, using Λ_3 with $\Lambda_3^\top \eta \Lambda_3 = \eta$, which is a Lorentz boost, $\Lambda_3^\top \Lambda_2^\top \Lambda_1^\top V \Lambda_1 \Lambda_2 \Lambda_3 = \Lambda_3^\top v \Lambda_3 = \tilde{v}_i \delta_{ij}$. Define $\tilde{\phi} \equiv \Lambda_3^\top \Lambda_2^\top \Lambda_1^\top \phi$, the edge action is now

$$S_{edge} = -\frac{1}{4\pi} \sum_I \int dtdx (\eta_I \partial_t + \tilde{v}_I \partial_x) \tilde{\phi}^I \partial_x \tilde{\phi}^I \tag{B.80}$$

The Hamiltonian is

$$H_{edge} = \frac{1}{4\pi}\sum_I \int dx \tilde{v}_I (\partial_x \phi^I)^2 \tag{B.81}$$

Here, $\partial_x \phi^I / 2\pi = \rho^I$ and, likewise, $\tilde{v}_I$ has to be positive in order for the energy to be positive. The direction of the propagation of the edge wave is dictated by $sgn(\eta_I)$. The thermal current is

$$J_T = \sum_I \eta_I \tilde{v}_I n_{T,I} = \int \frac{dq}{2\pi} \frac{\hbar \tilde{v}_I q}{e^{\hbar \tilde{v}_I q / k_b T} - 1} = \sum_I \eta_I \frac{\pi^2 k_B^2}{6h} T^2 \tag{B.82}$$

, where $n_{T,I}$ is the energy density of the mode I. Therefore, the thermal conductance $\kappa_T = \frac{\partial J_T}{\partial T}$ is

$$\kappa_T = \sum_I \eta_I \frac{\pi^2 k_B^2}{3h} T = \nu_T \kappa_0 \tag{B.83}$$

Here, we have defined $\nu_T = \sum_I \eta_I = Tr(\eta)$ and $\kappa_0 = \frac{\pi^2 k_B^2}{3h} T$ as the expression for the non-interacting system. The numbers of forward and backward going modes $\tilde{n}_\pm$ are given by $Tr\left(\frac{1 \pm \eta}{2}\right)$, respectively. Their difference is $\tilde{n}_+ - \tilde{n}_- = Tr(\eta)$. It is apparent that the thermal conductance of an edge mode contribute equallly regardless of its charge conductance.

Here, reconsider the relation between η and K. $K_D^{-1} = (\Lambda_1^\top K \Lambda_1)^{-1} = \Lambda_1^{-1} K^{-1} \Lambda_1^{\top -1}$, namely and $K^{-1} = \Lambda_1 K_D^{-1} \Lambda_1^\top$. Reversing the transformation $\Lambda_1 \Lambda_2 \Lambda_3 \eta \Lambda_3^\top \Lambda_2^\top \Lambda_1^\top = \Lambda_1 \Lambda_2 \eta \Lambda_2^\top \Lambda_1^\top = \Lambda_1 K_D^{-1} \Lambda_1^\top = K^{-1}$. Therfore, defining $M^\pm = \Lambda_1 \Lambda_2 \Lambda_3 \frac{1 \pm \eta}{2} \Lambda_3^\top \Lambda_2^\top \Lambda_1^\top$,

$$M^+ - M^- = K^{-1} \tag{B.84}$$

The difference between the number of forward-going modes and backward-going modes is given by the bulk topological matrix.

B.3.3 K-matrix and charge and thermal conductance

We have found above that both charge conductance $\sigma_H = \nu_C \frac{e^2}{h}$ and thermal conductance $\kappa_T = \nu_T \frac{\pi^2 k_B^2}{3h} T$ of fractional quantum Hall effects are quantized and dictated by the bulk topological property.

$$\nu_C = \mathbf{t}^\top K^{-1}\mathbf{t} \tag{B.85}$$

$$\nu_T = Tr(\eta) \tag{B.86}$$

As clearly seen, ν_C and ν_T are topological quantities. In order to keep ν_T invariant, one can only add counter-propagating edge channels in pairs, which does not affect the ν_C either.

The Wiedemann-Franz law states that the ratio of thermal conductivity over charge conductivity is proportional to temperature. The proportionality constant is called Lorentz number. Using the result of free electrons, $L_0 = \pi^2 k_B^2/3e^2$, the Lorentz number of fractional quantum Hall effects violates the free electron result.

$$L_H = \frac{\kappa_T}{T\sigma_H} = \frac{\nu_T}{\nu_C}L_0 \tag{B.87}$$

, which is reduced to the free electron result only in integer quantum Hall effects.

B.4 Two-terminal conductance of clean systems

B.4.1 Single channel

We now compute the two-terminal conductance of QHE states. Starting with $\nu = 1$, namely $K = 1$, $V = v$ and $\eta = 1$, where only single chiral electron channel is present. The charge density on the edge is $\rho = \frac{e}{2\pi}\partial_x\phi$. Regarding the continuity equation $\partial_t\rho + \partial_x I = 0$, $I = \frac{e}{2\pi}\partial_t\phi$. The action is

$$S = -\frac{\kappa}{4\pi}\int dtdx\partial_x\phi(i\partial_t + v\partial_x)\phi \tag{B.88}$$

Here, we use the imaginary time formalism. The DC response to an electrostatic potential $V(x')$ coupled to $\rho(x')$ is $\langle I\rangle = \int dtD(x-x',\omega\to 0)V(x')$.

$$\begin{aligned} D(x-x',\omega\to 0) &= -\int_{-\infty}^{0} dte^{-i\omega t}\frac{e^2}{(2\pi)^2\hbar}\langle[\partial_t\phi(x,0),\partial_x\phi(x',t)]\rangle \\ &= -\frac{e^2}{h}\sum_q e^{iq(x-x')}\frac{q\omega_n}{q(i\eta\omega_n - vq)} \\ &= \frac{e^2}{h}\theta(\eta(x-x'))\frac{i\eta\omega}{v}e^{i\eta(\omega+i\epsilon)(x-x')/v} \end{aligned} \tag{B.89}$$

In the step, the analytic continuation $i\omega_n \to \omega + i\epsilon$ was performed. The functional average was evaluated with S. The θ function reflects the chiral nature of the edge propagation, showing that the current at x depends only on the voltages at positions x' upstream of x. In the limit $\omega \to 0$, the integral will be dominated by values of x' that are deep into the upstream reservoir (the voltage in the left ohmic contact).

$$\langle I \rangle = \frac{e^2}{h}(V_L - V_R) \quad \Rightarrow \quad G_2 = \frac{e^2}{h} \tag{B.90}$$

Summing the contribution on the opposite edge $\langle I \rangle = -\frac{e^2}{h} V_R$ and compute the two-terminal conductance. Now let us see the effect of adding an intra-channel Coulomb interaction to the above simple model, namely $K = 1$ and $V = v_c$ as $\frac{1}{4\pi}\int dtdxv_c(\phi)^2$.

$$S = -\frac{1}{4\pi}\int dtdx\partial_x\phi(i\partial_t + (v + v_c)\partial_x)\phi \tag{B.91}$$

The effect of the Coulomb interaction is merely shifting the velocity, hence $G_2 = \frac{e^2}{h}$.

B.4.2 Co-propagating channels

The non-interacting $\nu = 2$ is merely doubling the single channel case, namely $K = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $V = \left(\begin{smallmatrix} v_1 & 0 \\ 0 & v_2 \end{smallmatrix}\right)$ and $\eta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which leads to $G_2 = 2\frac{e^2}{h}$. Hence let us add an inter-channel Coulomb interaction to $\nu = 2$, namely the velocity matrix is now $V = \left(\begin{smallmatrix} v_1 & v_{12} \\ v_{12} & v_2 \end{smallmatrix}\right)$.

$$\begin{aligned} S &= \frac{1}{4\pi}\int dtdx\Big[\partial_x\phi_1(i\partial_t + v_1\partial_x)\phi_1 + \partial_x\phi_2(i\partial_t + v_2\partial_x)\phi_2 + 2v_{12}\partial_x\phi_1\partial_x\phi_2\Big] \\ &= \frac{1}{4\pi}\sum_{q,\omega_n}\vec{\phi}^{\top *}_{q,\omega_n}\Omega\vec{\phi}_{q,\omega_n} \quad , \quad \Omega = q\begin{pmatrix} i\omega_n - v_1 q & v_{12}q \\ v_{12}q & i\omega_n - v_2 q \end{pmatrix} \end{aligned} \tag{B.92}$$

Here, $\vec{\phi}_{q,\omega_n} = (\phi_1(q,\omega_n), \phi_2(q,\omega_n))$. $det(\Omega) = 0$ yields the eigen velocities.

$$v_\pm = \frac{(v_1 + v_2) \pm \sqrt{(v_1 - v_2)^2 + 4v_{12}^2}}{2} \equiv \bar{v} \pm \frac{1}{2}\sqrt{\delta v^2 + 4v_{12}^2} \tag{B.93}$$

Therefore, $det(\Omega) = q(i\omega_n - v_+q)(i\omega_n - v_-q)$. The electric current is now $I = \frac{e}{2\pi}\partial_t(\phi_1 + \phi_2)$ and the correlator contains four terms. The functional average of each term corresponds to the component Ω^{-1}_{ij} for $i,j=1,2$. The two-terminal conductance can be derived from $D(x-x',\omega\to 0)=-\frac{e^2}{h}\sum_q e^{iq(x-x')}\sum_{ij}\Omega^{-1}_{ij}$. Defining $cos\theta = 2v_{12}/\sqrt{\delta v^2+4v_{12}^2}$,

$$\begin{aligned}\sum_{ij}\Omega^{-1}_{ij} &= \frac{i\omega_n - v_1 q + i\omega_n - v_2 q - 2v_{12}q}{(i\omega_n - v_+ q)(i\omega_n - v_- q)} \\ &= \frac{1+cos\theta}{i\omega_n - v_+ q} + \frac{1-cos\theta}{i\omega_n - v_- q} \\ \Rightarrow \quad G_2 &= (1+cos\theta+1-cos\theta)\frac{e^2}{h} \quad = \quad 2\frac{e^2}{h}\end{aligned} \tag{B.94}$$

The inter-channel Coulomb interaction in co-propagating channels has no effect on G_2.

B.4.3 Counter-propagating channels and KFP theory

We now consider $\nu = 2/3$, namely $K = \left(\begin{smallmatrix}1 & 0\\ 0 & -3\end{smallmatrix}\right)$, $V = \left(\begin{smallmatrix}v_1 & v_{12}\\ v_{12} & v_2\end{smallmatrix}\right)$ and $\eta = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$.

$$\begin{aligned}S &= \frac{1}{4\pi}\int dtdx\Big[\partial_x\phi_1(i\partial_t + v_1\partial_x)\phi_1 + 3\partial_x\phi_2(-i\partial_t + v_2\partial_x)\phi_2 + 2v_{12}\partial_x\phi_1\partial_x\phi_2\Big] \\ &= \frac{1}{4\pi}\sum_{q,\omega_n}\vec{\phi}^{\top *}_{q,\omega_n}\Omega\vec{\phi}_{q,\omega_n} \quad , \quad \Omega = q\begin{pmatrix} i\omega_n - v_1 q & v_{12}q \\ v_{12}q & -3i\omega_n - v_2 q\end{pmatrix}\end{aligned} \tag{B.95}$$

$det(\Omega)=0$ yields

$$v_\pm = \frac{(v_1-v_2)\pm\sqrt{(v_1+v_2)^2-4v_{12}^2/3}}{2} \tag{B.96}$$

Again the electric current is now $I = \frac{e}{2\pi}\partial_t(\phi_1+\phi_2)$.

$$\begin{aligned}3\sum_{ij}\Omega^{-1}_{ij} &= -\frac{i\omega_n - v_1 q - 3i\omega_n - 3v_2 q + 2v_{12}q}{(i\omega_n - v_+ q)(i\omega_n - v_- q)} \\ &= \frac{v_1+3v_2-2v_{12}+2v_+}{v_+-v_-}\frac{1}{i\omega_n - v_+ q} - \frac{v_1+3v_2-2v_{12}+2v_-}{v_+-v_-}\frac{1}{i\omega_n - v_- q} \\ &= \frac{\Delta+1}{i\omega_n - v_+ q} - \frac{\Delta-1}{i\omega_n - v_- q}\end{aligned}$$

We have introduced $c = (2v_{12}/\sqrt{3})/(v_1+v_2)$ and $\Delta = (2-\sqrt{3}c)/(\sqrt{1-c^2})$

$$G_2 = \frac{1}{3}(\Delta+1+\Delta-1)\frac{e^2}{h} = \frac{2}{3}\Delta\frac{e^2}{h} \tag{B.97}$$

Hence the inter-channel Coulomb interaction in counter-propagating channels makes G_2 non-universal and does not leads to the robust quantization of $G_2 = 2/3$ as observed in experiments. The stability criteria is $|c| < 1$. Only in the case of $c = \sqrt{3}/2$, $\Delta = 1$. For $v_{12} = 0$, $\Delta = 2$ giving rise to the maximum conductance.

We have considered two channels with different conducntances. We may also consider an imaginary $\nu = 1 - 1$, namely counter-propagating electron channels in one edge. $K = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $V = \left(\begin{smallmatrix} v_1 & v_{12} \\ v_{12} & v_2 \end{smallmatrix}\right)$ and $\eta = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. $det(\Omega) = 0$ yields the eigen velocities.

$$v_\pm = \frac{(v_1 - v_2) \pm \sqrt{(v_1 + v_2)^2 - 4v_{12}^2}}{2} \tag{B.98}$$

Again the electric current is now $I = \frac{e}{2\pi}\partial_t(\phi_1 + \phi_2)$.

$$\sum_{ij} \Omega_{ij}^{-1} = \frac{\Delta}{i\omega_n - v_+ q} - \frac{\Delta}{i\omega_n - v_- q} \tag{B.99}$$

, where $c = 2v_{12}/(v_1 + v_2)$ and $\Delta = (1 - c)/(\sqrt{1 - c^2})$ with $|c| < 1$.

$$G_2 = 2\Delta\frac{e^2}{h} \tag{B.100}$$

G_2 is again non-universal. The maximum conductance appears when $v_{12} = 0$ and $\Delta = 1$.

Here, I briefly summarize the flow of the theory by Kane, Fisher and Polchinski (KFP) [7]. As shown priory, the two terminal DC conductance of clean $\nu = 2/3$ state is $G_2 = 2\Delta\frac{e^2}{h}$. Only for $c = \sqrt{3}/2$, $\Delta = 1$. The non-universal conductance is due to the inter-channel Coulomb interaction between the counter-propagating channels. KFP [7] argued that the inclusion of the random inter-channel tunneling due to impurities is absolutely crucial for the robust DC conductance quantization as experimentally observed. Adding the random inter-channel tunneling S_{dis} and changing the basis to the "charge" and "spin" bases, the full action is now $S = S_c + S_\sigma + S_{dis,\sigma} + S_{c\sigma}$. If the coupling between the charge and spin modes $S_{c\sigma}$ can be neglected for the moment, the charge mode S_c is already quadratic and possesses a global $U(1)$ symmetry associated with the charge conservation. The disorder only affects the spin part $S_\sigma + S_{dis,\sigma}$. However, it turns out that a local $SU(2)$ transformation supplemented with an auxiliary field diagonalizes the "spin+disorder" part to S_n, possessing a global $SU(2)$ symmetry, which describes the upstream neutral mode propagating without any decay. Recovering the coupling term $S_{c\sigma}$, now the full action is $S = S_c + S_n + S_{cn}$, where $S_{c\sigma}$ is transformed to S_{cn} via the $SU(2)$ transformation. One can show that S_{cn} is an irrelevant operator upon renormalization transformation looking at the low energy. Namely, at the zero temperature, S_{cn} vanishes. Here, the random scattering is absolutely crucial for the coupling S_{cn} to be irrelevant. Therefore, at the zero-temperature (on the fixed line where $\Delta = 1$), the "disorder+Coulomb" problem is completely solved and the full action becomes merely $S = S_c + S_n$, where the charge and neutral modes are completely decoupled and both modes propagate infinite distance in each direction. At finite temperatures, the coupling term S_{cn} is still finite (not yet renormalized to zero), hence the neutral mode acquires a finite decay length. Employing the transformed action, the two-terminal conductance contributed solely by the charge mode is quantized as $G = \frac{2}{3}\frac{e^2}{h}$ even at finite temperatures.